7.4 | Determining Field from Potential

Learning Objectives

By the end of this section, you will be able to:

- Explain how to calculate the electric field in a system from the given potential
- Calculate the electric field in a given direction from a given potential
- Calculate the electric field throughout space from a given potential

Recall that we were able, in certain systems, to calculate the potential by integrating over the electric field. As you may already suspect, this means that we may calculate the electric field by taking derivatives of the potential, although going from a scalar to a vector quantity introduces some interesting wrinkles. We frequently need $\vec{E}$ to calculate the force in a system; since it is often simpler to calculate the potential directly, there are systems in which it is useful to calculate V and then derive $\vec{E}$ from it.

In general, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of $\vec{E}$ and also in the direction of lower potential V. Furthermore, the magnitude of $\vec{E}$ equals the rate of decrease of V with distance. The faster V decreases over distance, the greater the electric field. This gives us the following result.

Relationship between Voltage and Uniform Electric Field

In equation form, the relationship between voltage and uniform electric field is

$$E = -\frac{\Delta V}{\Delta s}$$

where Δs is the distance over which the change in potential ΔV takes place. The minus sign tells us that E points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

For continually changing potentials, ΔV and Δs become infinitesimals, and we need differential calculus to determine the electric field. As shown in **Figure 7.27**, if we treat the distance Δs as very small so that the electric field is essentially constant over it, we find that
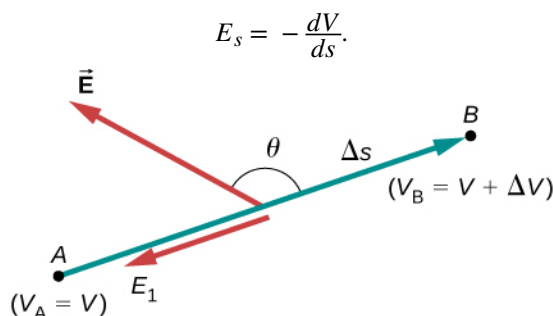
$$E_s = -\frac{dV}{ds}.$$

Figure 7.27 The electric field component along the displacement Δs is given by $E = -\dfrac{\Delta V}{\Delta s}$. Note that A and B are assumed to be so close together that the field is constant along Δs.

Therefore, the electric field components in the Cartesian directions are given by

$$E_x = -\frac{\partial V}{\partial x}, \ E_y = -\frac{\partial V}{\partial y}, \ E_z = -\frac{\partial V}{\partial z}. \tag{7.13}$$

This allows us to define the "grad" or "del" vector operator, which allows us to compute the gradient in one step. In Cartesian coordinates, it takes the form

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{7.14}$$

With this notation, we can calculate the electric field from the potential with

$$\vec{\mathbf{E}} = -\vec{\nabla} V, \tag{7.15}$$

a process we call calculating the gradient of the potential.

If we have a system with either cylindrical or spherical symmetry, we only need to use the del operator in the appropriate coordinates:

$$\text{Cylindrical: } \vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{7.16}$$

$$\text{Spherical: } \vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \tag{7.17}$$

Example 7.17

Electric Field of a Point Charge

Calculate the electric field of a point charge from the potential.

Strategy

The potential is known to be $V = k\frac{q}{r}$, which has a spherical symmetry. Therefore, we use the spherical del operator in the formula $\vec{\mathbf{E}} = -\vec{\nabla} V$.

Solution

Performing this calculation gives us

$$\vec{\mathbf{E}} = -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)k\frac{q}{r} = -kq\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{1}{r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{1}{r} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\frac{1}{r}\right).$$

This equation simplifies to

$$\vec{\mathbf{E}} = -kq\left(\hat{\mathbf{r}}\frac{-1}{r^2} + \hat{\boldsymbol{\theta}}0 + \hat{\boldsymbol{\varphi}}0\right) = k\frac{q}{r^2}\hat{\mathbf{r}}$$

as expected.

Significance

We not only obtained the equation for the electric field of a point particle that we've seen before, we also have a demonstration that $\overrightarrow{\mathbf{E}}$ points in the direction of decreasing potential, as shown in **Figure 7.28**.
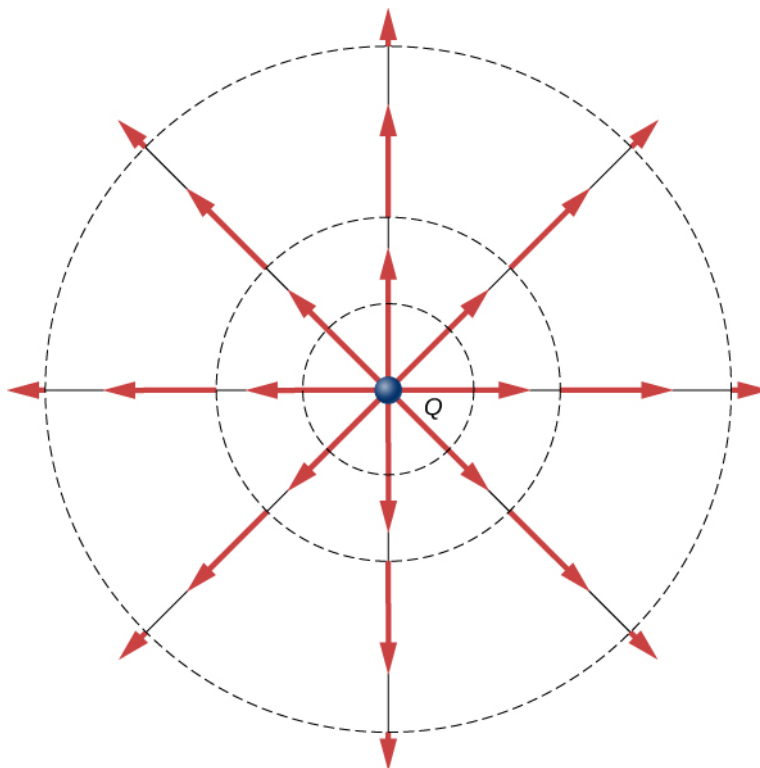

Figure 7.28   Electric field vectors inside and outside a uniformly charged sphere.

Example 7.18

Electric Field of a Ring of Charge

Use the potential found in **Example 7.8** to calculate the electric field along the axis of a ring of charge (**Figure 7.29**).
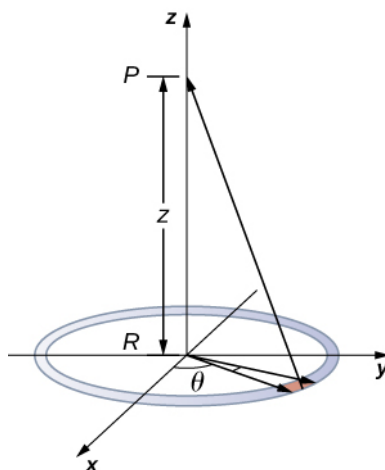
Figure 7.29 We want to calculate the electric field from the electric potential due to a ring charge.

Strategy

In this case, we are only interested in one dimension, the *z*-axis. Therefore, we use $E_z = -\frac{\partial V}{\partial z}$

with the potential $V = k\frac{q_{tot}}{\sqrt{z^2 + R^2}}$ found previously.

Solution

Taking the derivative of the potential yields

$$E_z = -\frac{\partial}{\partial z}\frac{kq_{tot}}{\sqrt{z^2 + R^2}} = k\frac{q_{tot}\,z}{\left(z^2 + R^2\right)^{3/2}}.$$

Significance

Again, this matches the equation for the electric field found previously. It also demonstrates a system in which using the full del operator is not necessary.

7.11 Check Your Understanding Which coordinate system would you use to calculate the electric field of a dipole?

7.5 | Equipotential Surfaces and Conductors

Learning Objectives

By the end of this section, you will be able to:

- Define equipotential surfaces and equipotential lines
- Explain the relationship between equipotential lines and electric field lines
- Map equipotential lines for one or two point charges
- Describe the potential of a conductor
- Compare and contrast equipotential lines and elevation lines on topographic maps

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. This is not